

# Hamiltonian description of self-consistent wave-particle dynamics in a periodic structure

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The coupled dynamics of electrons and electromagnetic fields propagating in traveling wave tubes is expressed with a hamiltonian formulation. The field is represented with eigenfunctions adapted to Floquet boundary conditions along the tube axis, using the Gel'fand  $\beta$ -transform. The electron hamiltonian is the standard one coupling the particles to the propagating fields. The dynamics conserves energy, and excludes self-acceleration. A complete hamiltonian formulation of the dynamics results from adding space charge effects by electrostatic action-at-a-distance coupling.

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To model consistently the interaction between electrons and waves in devices such as the traveling wave tube, free electron laser or synchrotron, one is presently left with two options. The first one [4, 9, 15] is to consider the flow of electrons as a distributed charge and current density coupled with the field through the Maxwell equations. Since it generates diverging singularities, the particle nature of electrons is intentionally overlooked until the question of determining the trajectories of the flow is raised. Here the only possibility is to switch back to a particle description in which the Lorentz force applies. This change of model for the flow precludes the description of the wave-electron system in hamiltonian form. One reason is that a procedure is needed to distribute the electron charge and current into a finite volume. This procedure, usually based on meshing space, can only be arbitrary. The second option [6, 8, 11] is to consistently consider electrons as particles and to determine the field they radiate starting from the Liénard-Wiechert potentials. The problem is that it is not possible to calculate the reaction from the radiated field because all quantities are diverging at the electron position. Attempts to balance energy and momentum resorting to *a posteriori* forces such as the Abraham-Lorentz force are plagued with a number of difficulties [8, 11, 14, 16]. Among them the infinite rest mass of the electron, self-acceleration and acausality, are also incompatible with the existence of a well-posed hamiltonian.

Hamiltonians are essential to consistently define energy and momentum. They also have great practical usefulness to find approximate solutions in complex systems or to control errors in numerical integration schemes [5]. So their absence in the case of the classical wave-electron interaction is both theoretically and practically unsatisfactory.

While the final solution to these problems may involve an (upgraded, regular) quantum electrodynamics, one could at least expect a consistent classical approximation, which would be the classical limit of its quantum counterpart. For instance, in the limit where the electron radiates a large number of photons, each of them having a small energy compared to its kinetic energy, which is the case for the aforementioned devices, one could expect that averaging the quantum theory over a large number of emitted photons would provide a consistent classical limit. But such a theory, free of the above problems, remains to be found [7, 14]. To progress in this direction, we undertake the construction of a hamiltonian for the traveling wave tube.

First of all, one can distinguish the space-charge field, which is the curl-free part of the fields, from the radiated field, which is the divergence-free part. In free space, the space-charge field between several electrons with positions  $\mathbf{r}_i$  derives from the hamiltonian  $\mathcal{H}_{sc} = \frac{1}{8\pi\epsilon_0} \sum_{i \neq j} e^2 / \|\mathbf{r}_i - \mathbf{r}_j\|$ . Adopting this point of view eliminates the problem of infinite rest mass, by eliminating altogether the electrostatic field degrees of freedom. So one is left with finding a hamiltonian for the radiated fields coupled with the electrons. This is the object of this work.

The traveling wave tube is a waveguide, periodic along the  $x$  axis with period  $d$ . All equations are written in the waveguide reference frame. It is bounded laterally by perfect metal ( $\mathbf{E} \times \mathbf{e}_\perp = \mathbf{0}$ ,  $\mathbf{H} \cdot \mathbf{e}_\perp = 0$ ) boundary condition. All periods, with shape  $\mathcal{V}_0$  and volume  $|\mathcal{V}_0|$ , form a domain  $\mathcal{V}_\mathbb{Z} := \cup_{n \in \mathbb{Z}} \{\mathbf{r} + n d \mathbf{e}_x : \mathbf{r} \in \mathcal{V}_0\}$ . The waves propagating in the tube are described with their electric and magnetic fields,  $\mathbf{E}(\mathbf{r}, t)$ ,  $\mathbf{H}(\mathbf{r}, t)$ , obeying the Maxwell equations with sources inside the tube [1].

We decompose the radiated field on the function ba-

sis made of the propagating modes. These modes satisfy the lateral boundary condition and the Floquet condition,  $\mathbf{E}(\mathbf{r} + d\mathbf{e}_x) = e^{-i\beta d}\mathbf{E}(\mathbf{r})$  for some  $\beta \in \mathbb{R}$ . Inside  $\mathcal{V}_0$ , they satisfy  $\nabla \cdot \mathbf{E}_{s\beta} = 0, \nabla \cdot \mathbf{H}_{s\beta} = 0$ , and

$$\text{rot } \mathbf{E}_{s\beta} = -i\mu_0\Omega_{s\beta}\mathbf{H}_{s\beta} \quad (1)$$

$$\text{rot } \mathbf{H}_{s\beta} = i\epsilon_0\Omega_{s\beta}\mathbf{E}_{s\beta} \quad (2)$$

where the  $\Omega_{s\beta}$  are the real eigenvalues,  $\mu_0$  and  $\epsilon_0$  the permeability and permittivity of vacuum. For a given  $\beta \in [0, 2\pi/d]$ , the set of eigen electric fields  $\mathbf{E}_{s\beta}$  ( $s \in \mathbb{N}$ ) is an orthogonal basis of the divergence-free subset of  $H(\text{rot}, \mathcal{V}_0)$  (this is the Hilbert space of square-integrable fields on  $\mathcal{V}_0$  which have a square-integrable curl). Any divergence-free field  $\mathbf{E}_\beta(\mathbf{r}, t)$  on  $\mathcal{V}_0$  can be decomposed on the  $\mathbf{E}_{s\beta}$  with coefficients  $V_{s\beta}$ ,

$$\mathbf{E}_\beta(\mathbf{r}, t) = \sum_{s \in \mathbb{N}} V_{s\beta}(t) \mathbf{E}_{s\beta}(\mathbf{r}). \quad (3)$$

This decomposition can be continued to  $\mathcal{V}_\mathbb{Z}$  thanks to the Floquet condition, and the continuation field satisfies the Floquet condition as well (possibly with a Gibbs phenomenon, depending on the field regularity).

An arbitrary function  $G$  in the propagating structure does not, in general, satisfy the Floquet condition. But it can be expanded on functions  $G_\beta$  satisfying the Floquet condition. Indeed, the Fourier series on the variable  $\beta$ , with  $x$  as a parameter,

$$G_\beta(x, t) := \sum_{n \in \mathbb{Z}} G(x + nd, t) e^{in\beta d} \quad (4)$$

satisfies the Floquet condition with respect to  $x$ . The  $n$ -th Fourier coefficient in the series is given by

$$G(x + nd, t) = \frac{1}{2\pi} \int_{\beta d=0}^{2\pi} G_\beta(x, t) e^{-jn\beta d} d(\beta d), \quad (5)$$

which for  $n = 0$  yields exactly the requested expansion. The transform (4), hereafter referred to as the  $\beta$ -transform [3, 10, 13, 17], decomposes any function into a superposition of functions satisfying the Floquet condition. It will be central in what follows. Relation (5) is the inverse  $\beta$ -transform.

Using equation (3), we finally get

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{2\pi} \sum_{s \in \mathbb{N}} \int_{\beta d=0}^{2\pi} V_{s\beta}(t) \mathbf{E}_{s\beta}(\mathbf{r}) d(\beta d). \quad (6)$$

This decomposition holds for any free electric field in the structure, e.g. propagating modes, evanescent modes or superposition of these. The magnetic field expansion is written

$$\mathbf{H}_\beta(\mathbf{r}, t) = i \sum_{s \in \mathbb{N}} I_{s\beta}(t) \mathbf{H}_{s\beta}(\mathbf{r}) \quad (7)$$

with the factor  $i$  for later convenience.

Our initial choice to consider independently the propagating field holds only if it is decoupled from the space-charge field. This latter adds the curl-free term  $-\nabla\phi$  to

the expression of  $\mathbf{E}$  where  $\phi$  satisfies the Poisson equation  $\Delta\phi = -\rho/\epsilon_0$ . The  $\beta$ -transform commutes both with time and space derivatives, so Maxwell equations are valid for  $\mathbf{E}_\beta$  and  $\mathbf{H}_\beta$  as well. Using the  $\beta$ -transform of  $\phi$ , the general form of the electric field in presence of charged particles is

$$\mathbf{E}_\beta(\mathbf{r}, t) = \sum_{s \in \mathbb{N}} V_{s\beta}(t) \mathbf{E}_{s\beta}(\mathbf{r}) - \nabla\phi_\beta. \quad (8)$$

Using Maxwell equations, the field decompositions (8) and (7), and the definition of eigenfields  $\mathbf{E}_{s\beta}$  and  $\mathbf{H}_{s\beta}$ , we get

$$-\sum_s I_{s\beta} \Omega_{s\beta} \mathbf{E}_{s\beta} = \sum_s \dot{V}_{s\beta} \mathbf{E}_{s\beta} + \frac{\mathbf{J}_\beta}{\epsilon_0} - \frac{\partial \nabla\phi_\beta}{\partial t}, \quad (9)$$

$$\sum_s V_{s\beta} \Omega_{s\beta} \mathbf{H}_{s\beta} = \sum_s \dot{I}_{s\beta} \mathbf{H}_{s\beta}, \quad (10)$$

where  $\mathbf{J}$  is the current density. We multiply the first expression by the conjugate of the field  $\mathbf{E}_{s'\beta}^*$  and the second expression by  $\mathbf{H}_{s'\beta}^*$ . Integrating the resulting expressions over the volume  $\mathcal{V}_0$  yields vanishing terms for  $s \neq s'$ . Otherwise we exhibit the electric  $\frac{1}{2}\epsilon_0 \int_{\mathcal{V}_0} |\mathbf{E}_{s\beta}|^2 d^3\mathbf{r}$  and magnetic  $\frac{1}{2}\mu_0 \int_{\mathcal{V}_0} |\mathbf{H}_{s\beta}|^2 d^3\mathbf{r}$  energies stored in one period for the corresponding propagation mode. Both energies are equal to half the total stored energy  $N_{s\beta}$  of the basis function. The volume integral involving  $\nabla\phi_\beta$  is transformed to a surface integral thanks to the identity  $\nabla \cdot (\phi_\beta \mathbf{E}_{s'\beta}^*) = (\nabla\phi_\beta) \cdot \mathbf{E}_{s'\beta}^* + \phi_\beta \nabla \cdot \mathbf{E}_{s'\beta}^*$ . The electric field basis functions are divergence-free, so only the surface integral  $\int_{\partial\mathcal{V}_0} \phi_\beta \mathbf{E}_{s'\beta}^* \cdot \mathbf{e}_\perp dS$  remains. The surface integral over the two cross-sections of the waveguide vanishes because both  $\phi_\beta$  and  $\mathbf{E}_{s'\beta}$  satisfy the Floquet condition. The lateral metallic parts are at imposed time-invariant potentials, a property  $\phi_\beta$  inherits, so the time derivative of the corresponding surface integral vanishes. This concludes our demonstration that the evolution of the  $V_{s\beta}$  and  $I_{s\beta}$  representing the propagating field is decoupled from the space-charge field

$$\dot{V}_{s\beta} + \Omega_{s\beta} I_{s\beta} = -\frac{1}{N_{s\beta}} \int_{\mathcal{V}_0} \mathbf{J}_\beta \cdot \mathbf{E}_{s\beta}^* d^3\mathbf{r}, \quad (11)$$

$$\dot{I}_{s\beta} - \Omega_{s\beta} V_{s\beta} = 0. \quad (12)$$

We perform the inverse  $\beta$ -transform of (11) and (12). Like the usual Fourier coefficients, the  $\beta$ -transform of a product is the convolution of its factor transforms. The source term in Maxwell-Ampère is transformed by (4) into an integral over the complete volume  $\mathcal{V}_\mathbb{Z}$ :

$$\begin{aligned} \int_{\mathcal{V}_0} \mathbf{J}_\beta \cdot \mathbf{E}_{s\beta}^* d^3\mathbf{r} &= \int_{\mathcal{V}_0} \sum_n \mathbf{J}(\mathbf{r} + nd\mathbf{e}_x) \cdot \mathbf{E}_{s\beta}^*(\mathbf{r}) e^{in\beta d} d^3\mathbf{r} \\ &= \int_{\mathcal{V}_0} \sum_n \mathbf{J}(\mathbf{r} + nd\mathbf{e}_x) \cdot \mathbf{E}_{s\beta}^*(\mathbf{r} + nd\mathbf{e}_x) d^3\mathbf{r} \\ &= \int_{\mathcal{V}_\mathbb{Z}} \mathbf{J} \cdot \mathbf{E}_{s\beta}^* d^3\mathbf{r}, \end{aligned}$$

where the second equality follows from the Floquet property of  $\mathbf{E}_{s\beta}$ . Finally, using the inverse  $\beta$ -transform,

$$\dot{V}_{sn} + \sum_m \Omega_{sm} I_{s,n-m} = - \int_{V_z} \mathbf{J} \cdot \mathbf{F}_{s,-n} d^3\mathbf{r}, \quad (13)$$

$$\dot{I}_{sn} - \sum_m \Omega_{sm} V_{s,n-m} = 0, \quad (14)$$

where

$$\mathbf{F}_{s,-n} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\mathbf{E}_{s\beta}^*}{N_{s\beta}} e^{in\beta d} d(\beta d) \quad (15)$$

whose  $\beta$ -transform is  $\mathbf{F}_{s\beta} = \mathbf{E}_{s\beta}/N_{s\beta}$ .

The electric and magnetic fields, given by

$$\mathbf{E}(\mathbf{r}, t) = \sum_{s,n} V_{sn}(t) \mathbf{E}_{s,-n}(\mathbf{r}) - \nabla \phi(\mathbf{r}, t), \quad (16)$$

$$\mathbf{H}(\mathbf{r}, t) = i \sum_{s,n} I_{sn}(t) \mathbf{H}_{s,-n}(\mathbf{r}), \quad (17)$$

are rewritten

$$\mathbf{E}(\mathbf{r}, t) = \sum_{s,n} V_{sn}(t) \mathbf{E}_{s0}(\mathbf{r} - nd\mathbf{e}_x) - \nabla \phi(\mathbf{r}, t), \quad (18)$$

$$\mathbf{H}(\mathbf{r}, t) = i \sum_{s,n} I_{sn}(t) \mathbf{H}_{s0}(\mathbf{r} - nd\mathbf{e}_x). \quad (19)$$

Each eigenfunction  $\mathbf{H}_{s\beta}$  derives from a vector potential  $\mathbf{A}_{s\beta}$

$$\mu_0 \mathbf{H}_{s\beta} = \text{rot } \mathbf{A}_{s\beta}. \quad (20)$$

The inverse  $\beta$ -transform yields  $\mu_0 \mathbf{H}_{sn} = \text{rot } \mathbf{A}_{sn}$ . Using (7), we find

$$\mathbf{A} = i \sum_{s,n} I_{sn} \mathbf{A}_{s,-n} + \nabla u \quad (21)$$

where  $u$  is any function of space corresponding to a gauge choice.

These expressions offer an interpretation for the index  $n$ . The electric field at a particular position is the superposition of the field shape  $\mathbf{E}_{s0}$  generated by the adjacent cells of the periodic structure with amplitudes  $V_{sn}$ . Therefore  $V_{sn}$  (resp.  $I_{sn}$ ) can be seen as the contribution of the  $s$ -th eigenmode in cell  $n$  to the electric (resp. magnetic) field. The propagation is described by linearly coupled harmonic oscillators. Cell  $n$  is coupled with cell  $m$  through the coefficient  $\Omega_{s,n-m} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Omega_{s\beta} e^{i(n-m)\beta d} d(\beta d)$ , viz. the corresponding Fourier coefficient of the dispersion curve  $\Omega_{s\beta}$ . The reciprocity condition  $\Omega_{s,-\beta} = \Omega_{s\beta}$  ensures  $\Omega_{s,-n} = \Omega_{sn} \in \mathbb{R}^+$ . The same condition on the fields reads  $\mathbf{E}_{s,-\beta} = \mathbf{E}_{s\beta}^*$ , therefore  $\mathbf{E}_{sn} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \mathbf{E}_{s\beta}^* e^{in\beta d} d(\beta d) = \frac{1}{\pi} \Re \left( \int_0^{+\pi} \mathbf{E}_{s\beta}^* e^{in\beta d} d(\beta d) \right)$  is a vector with real components only. Symmetrically,  $i\mathbf{H}_{sn}$  has only real components. In summary,  $V_{sn}$ ,  $I_{sn}$ ,  $\mathbf{E}_{sn}$ ,  $\mathbf{F}_{sn}$ ,  $i\mathbf{H}_{sn}$ ,  $i\mathbf{A}_{sn}$ ,  $\Omega_{sn}$ ,  $N_{sn}$  are all real quantities.

Starting from this model, we can now construct a hamiltonian for the electromagnetic field. The standard [6] hamiltonian  $\frac{1}{2} \int_V (\epsilon_0 |\mathbf{E}|^2 + \mu_0 |\mathbf{H}|^2) d^3\mathbf{r}$  results from the Lorentz force and the Maxwell-Ampère equation. In this work, we want to make no assumption on the final form of the electromagnetic hamiltonian because corrective terms might be needed. So we cannot use it as a starting point. Instead, the previous development provides a hamiltonian for the fields alone, independently from how they are coupled with the electrons. The vectors  $V_s = (\dots V_{sn} \dots)$  and  $I_s = (\dots I_{sn} \dots)$  represent the state of the fields, and their time evolutions derive from the hamiltonian (with conjugate variables  $V, I$ )

$$\mathcal{H}_{\text{em}}(V_s, I_s)_{s \in \mathbb{N}} = \frac{1}{2} \sum_s (V_s Q_s V_s^t + I_s Q_s I_s^t), \quad (22)$$

where  $\cdot^t$  denotes vector transpose, and  $Q_s$  is the infinite matrix with entries  $(Q_s)_{n,m} = \Omega_{s,n-m}$  for  $(n, m) \in \mathbb{Z}^2$ . Till now, the basis functions  $\mathbf{E}_{s\beta}$  and  $\mathbf{H}_{s\beta}$  can have arbitrary physical dimensions. But adopting (22) as a hamiltonian results in  $V_s Q_s V_s^t$  and  $I_s Q_s I_s^t$  being energies. Hence  $V_s$  and  $I_s$  have the same dimensions. Equations (13)-(14) impose  $\Omega_{sn}$  to be a frequency. Therefore  $V_s$  and  $I_s$  have the dimension of the square root of an action (energy divided by frequency). The electric and magnetic field have their standard dimensions and  $N_{s\beta}$  has the dimension of a frequency.

The hamiltonian of the electron in the field of a given 4-potential  $(\phi, \mathbf{A})$  is [11]

$$\mathcal{H}_{\text{el}}(\mathbf{p}, \mathbf{r}) = \sqrt{m^2 c^4 + c^2 |\mathbf{p} - e\mathbf{A}(\mathbf{r})|^2} + e\phi(\mathbf{r}) \quad (23)$$

where  $e$  and  $m$  are the electron charge and mass,  $c$  is the celerity of light, and the momentum  $\mathbf{p}$  is the canonical conjugate of the electron position  $\mathbf{r}$ . The complete hamiltonian for the fields and  $M$  electrons is necessarily the sum of the purely electromagnetic hamiltonian  $\mathcal{H}_{\text{em}}$  with the Lorentz force dynamic hamiltonian  $\mathcal{H}_{\text{el}}$ , where  $\mathbf{A}$  is a function of the  $I_{sn}$  as given by (21) and  $\phi$  generates the space-charge field,

$$\mathcal{H}(\mathbf{p}_j, \mathbf{r}_j, V_s, I_s)_{\{1 \leq j \leq M, s \in \mathbb{N}\}} = \sum_j \mathcal{H}_{\text{el}}(\mathbf{r}_j, \mathbf{p}_j) + \mathcal{H}_{\text{em}}. \quad (24)$$

The complete physics of the system is governed by it, including the source term in the Maxwell-Ampère equation. This source term has not been used to build the hamiltonian and is equivalent to equation (13). In presence of a single electron with position  $\mathbf{r}(t)$  and velocity  $\mathbf{v}(t)$ , we have [11, §29]  $\mathbf{J}(\mathbf{r}, t) = e\mathbf{v}(t) \delta(\mathbf{r} - \mathbf{r}(t))$ , and (13) simplifies into

$$\dot{V}_{sn} = -e\mathbf{v} \cdot \mathbf{F}_{s,-n}(\mathbf{r}) - \sum_m \Omega_{sm} I_{s,n-m}. \quad (25)$$

Its derivation from  $\mathcal{H}$  is obtained considering that  $V_{sn}$  is the generalized momentum canonically conjugate to the gener-

alized coordinate  $I_{sn}$ ,

$$\begin{aligned}\dot{V}_{sn} &= -\frac{\partial \mathcal{H}}{\partial I_{sn}} \\ &= -\frac{\partial \mathcal{H}_{em}}{\partial I_{sn}} + \frac{\mathbf{p} - e\mathbf{A}}{\sqrt{m^2 c^4 + c^2 |\mathbf{p} - e\mathbf{A}|^2}} \cdot e i \mathbf{A}_{s,-n} \\ &= e \mathbf{v} \cdot i \mathbf{A}_{s,-n}(\mathbf{r}) - \sum_m \Omega_{sm} I_{s,n-m}\end{aligned}\quad (26)$$

Here we used the relation  $\mathbf{v} = \dot{\mathbf{r}} = \partial \mathcal{H}_{el} / \partial \mathbf{p} = c^2 (\mathbf{p} - e\mathbf{A}) / \sqrt{m^2 c^4 + c^2 |\mathbf{p} - e\mathbf{A}|^2}$ , that derives directly from the electron hamiltonian (23).

Equations (25) and (26) are incompatible unless

$$\mathbf{F}_{sn}(\mathbf{r}) = -i \mathbf{A}_{sn}(\mathbf{r}). \quad (27)$$

The  $\beta$ -transform gives  $\mathbf{E}_{s\beta} / N_{s\beta} = \mathbf{F}_{s\beta} = -i \mathbf{A}_{s\beta}$ . By definition,  $\mathbf{E}_{s\beta} = -i \Omega_{s\beta} \mathbf{A}_{s\beta} - \nabla u_{s\beta}$ , so the two equations reconcile when

$$N_{s\beta} = \Omega_{s\beta} \quad \text{and} \quad \nabla u_{s\beta} = 0. \quad (28)$$

It is always possible to choose basis functions satisfying the Coulomb gauge, which is the second condition. The first condition clarifies the question of the electromagnetic energy. In this work, it is given by the hamiltonian  $\mathcal{H}_{em}$  while the usual Poynting energy is  $\frac{1}{2} \int_{V_z} (\epsilon_0 |\mathbf{E}|^2 + \mu_0 |\mathbf{H}|^2) d^3 \mathbf{r}$ . In the latter expression, consider the electric part  $\mathcal{E} = \frac{1}{2} \epsilon_0 \int_{V_z} |\mathbf{E}|^2 d^3 \mathbf{r} = \frac{1}{2} \epsilon_0 \sum_n \int_{V_0} |\mathbf{E}(\mathbf{r} + n d \mathbf{e}_x)|^2 d^3 \mathbf{r}$ . Using Parseval's relation, this is also  $\frac{1}{2} \epsilon_0 \int_{V_0} \int_0^{2\pi} |\mathbf{E}_\beta|^2 d(\beta d) d^3 \mathbf{r}$ . Now, using the decomposition (8) of the electric field, we find  $\mathcal{E} = \frac{1}{2} \epsilon_0 \int_{V_0} \int_0^{2\pi} |\sum_s V_{s\beta} \mathbf{E}_{s\beta}|^2 d(\beta d) d^3 \mathbf{r}$ . Orthogonality of the eigenmodes reduces this expression to  $\mathcal{E} = \frac{1}{2} \sum_s \int_0^{2\pi} N_{s\beta} |V_{s\beta}|^2 d(\beta d)$ . A similar calculation applies to the magnetic energy. Finally, using the inverse  $\beta$ -transform and the condition  $N_{s\beta} = \Omega_{s\beta}$ , we find that both expressions for the energy,  $\mathcal{H}_{em}$  and Poynting's, are equal :

$$\frac{1}{2} \int_{V_z} (\epsilon_0 |\mathbf{E}|^2 + \mu_0 |\mathbf{H}|^2) d^3 \mathbf{r} = \frac{1}{2} \sum_s (V_s Q_s V_s^t + I_s Q_s I_s^t). \quad (29)$$

In this work, we constructed a hamiltonian describing the electrons coupled with the electromagnetic fields propagating in a periodic structure. This coupled oscillator model may be viewed as a generalization of the telegraph delay line where each period consists of one inductor and capacitor, coupled with their nearest neighbours and electrons [12]. In contrast, in the present work all periods are coupled with each other. This model enables computing the time-dependent behaviour of a helix traveling-wave tube, faster than industrial PIC codes [2].

While we focused on divergence-free, propagating fields, the space charge field may be taken into account by adding the "space charge hamiltonian" of two or more interacting electrons or by adding Darwin's first-order relativistic approximation [8, §12.6] (Green functions are

needed to take into account the waveguide boundary condition). In this action-at-a-distance approach with two electrons interacting through their electrostatic field, the problem of infinite electromagnetic mass does not occur. This hamiltonian is formally independent of time but depends explicitly on space through the  $\mathbf{A}_{sn}(\mathbf{r})$ . Indeed the boundary conditions – for example a corrugated metallic wall – enable the field to exchange momentum but not energy. Maxwell's equations and the Lorentz force correctly account for the recoil from the radiated field. This hamiltonian is the sum of two unconditionally positive contributions which are minimum when the system is at rest, a situation precluding self-acceleration. As a consequence of the hamiltonian, the source term in the equation of Maxwell-Ampère and the Lorentz force consistently express how the electron is coupled with the propagating field.

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